N-body-extended Channel Estimation for Low-Noise Parameters

M. Hotta*, ^{1} T. Karasawa§, ^{2} and M. Ozawa§ ^{3}

* *Department of Physics, Faculty of Science, Tohoku University,*
Sendai, 980-8578, Japan

§ *Graduate School of Information Sciences, Tohoku University,*
Sendai, 980-8579, Japan

Abstract

The notion of low-noise channels was recently proposed and analyzed in detail in order to describe noise-processes driven by environment [M. Hotta, T. Karasawa and M. Ozawa, Phys. Rev. **A72**, 052334 (2005)]. An estimation theory of low-noise parameters of channels has also been developed. In this report, we address the low-noise parameter estimation problem for the N-body extension of low-noise channels. We perturbatively calculate the Fisher information of the output states in order to evaluate the lower-bound of the mean-square error of the parameter estimation. We show that the maximum of the Fisher information over all input states can be attained by a factorized input state in the leading order of the low-noise parameter. Thus, to achieve optimal estimation, it is not necessary for there to be entanglement of the N subsystems, as long as the true low-noise parameter is sufficiently small.

^{1}hotta@tuhep.phys.tohoku.ac.jp
^{2}jidai@ims.is.tohoku.ac.jp
^{3}ozawa@math.is.tohoku.ac.jp

1   Introduction

Quantum channels or quantum operations [1] can be used to describe low-noise processes of physical systems coupled weakly with the environment, instead of invoking complicated Hamiltonians. We have introduced the notion of low-noise quantum channels Γ_ϵ characterized by one low-noise parameter ϵ in an earlier paper [2]. Low-noise quantum channels are very useful for many physical applications, including relaxation processes driven by a thermal bath, decoherence of quantum computers, and rare processes in elementary particle physics.

Determining of the order of the low-noise parameter ϵ is often crucial in various fields. For example, in elementary particle physics, low-noise signals may be generated by some new physical processes and the source characterization of the low noise can give us information on the structure of the new physics theory [3]. In developing scalable quantum computers, it is also important to determine the order of low noise, as it must be eliminated to maintain quantum coherence.

In Ref. [2], we present an estimation theory for noise parameters in low-noise quantum channels, which we briefly review here. A fundamental quantity in the theory is the Fisher information of the output state of the channel Γ_ϵ. Consider an input state ρ_{in} for the output state $\rho = \rho_{out}(\epsilon)$, obtained by

$$\rho_{out}(\epsilon) = \Gamma_\epsilon[\rho_{in}]. \tag{1}$$

The symmetric logarithmic derivative L is defined by

$$\partial_\epsilon \rho = \frac{1}{2}\left(L\rho + \rho L\right), \ L^\dagger = L. \tag{2}$$

The Fisher information of the output state is defined by

$$J = \mathrm{Tr}[\rho L^2]. \tag{3}$$

It is well known that the inverse of J is the lower bound of the mean-square error of unbiased estimators [4, 5]. The maximum of J over all the input states can be attained from a pure input state [6]. Hence, in later discussions, we focus on pure input states. The Fisher information of output states

for an ancilla-extension of the low-noise channel $\Gamma_\epsilon \otimes id_A$ was calculated, taking account of entanglement between the original system S and the ancilla system A. For a qubit system, a characteristic parameter was defined associated with a general low-noise channel, and the channels for which the prior entanglement increases the output Fisher information in terms of the range of that parameter were characterized. The optimal input pure states were discussed for general low-noise channels Γ_ϵ. We introduced an enhancement factor, representing the ratio of the Fisher information of the ancilla-assisted estimation to that of the original system, and showed that it is always upper bounded by 3/2.

We define here an N-body-extended channel. We take N identical systems, with the state space given by $\mathcal{H}^{\otimes N}$, where $\mathcal{H}$ is the state space of the original system. Suppose that a quantum channel Γ_θ with an unknown parameter θ acts on a state of $\mathcal{H}$. The N-body extension of Γ_θ is defined by $\Gamma_\theta^{\otimes N}$. The estimation problem of $\Gamma_\theta^{\otimes N}$ is to find the optimal output measurements and input states. This N-body-extended problem is nontrivial. Using collective measurements of the composite system and entangled input states, the problem cannot be simply reduced to the original estimation problem of Γ_θ. Solutions for optimizing the input states can be obtained only by specific models [7] [13], and a complete solution to the problem remains to be determined. For unitary channel estimations [13], the optimal input states of the estimation are pure states strongly entangled among the N subsystems. Compared with factorized input states, difficult controls are required to set up such entangled states in real experiments. If optimization by factorized input states is possible for a specific class of channels, the physical realization of the entangled input states for the channels is not important. For example, it is known that factorized-input-state optimization is satisfied for a generalized Pauli channel [7].

In this paper, we discuss the estimation theory of an N-body extended low-noise channel $\Gamma_\epsilon^{\otimes N}$, using its ancilla-extension. The ancilla-extended channel is given by $\Gamma_\epsilon^{\otimes N} \otimes id$, where id is the identical channel of the ancilla system. As pointed out in Ref. [2], in order to maximize the output Fisher information, it is sufficient to adopt an ancilla state space with dimensions equal to that of the object system. Therefore, we can assume that the ancilla state space is decomposed into N identical spaces. By taking an ancilla-extended low-noise channel $\Gamma_\epsilon \otimes id_A$, as analyzed in Ref. [2], the

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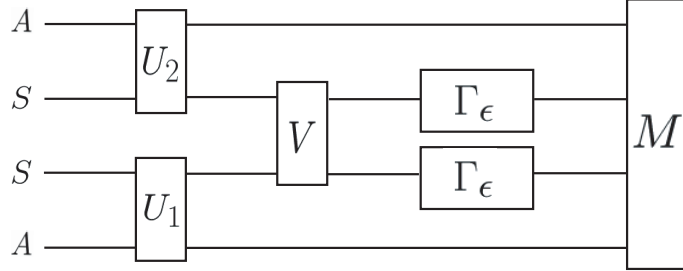
Figure 1: Example of $(\Gamma_\epsilon \otimes id_A)^{\otimes N}$ with $N = 2$. The initial state is transformed into an entangled state by unitary operators U_1, U_2, and V. The non-ancilla subsystems go through a low-noise channel Γ_ϵ. A collective measurement M is made of the output state to estimate the low-noise parameter ϵ.

channel $\Gamma_\epsilon^{\otimes N} \otimes id$ can be regarded as $(\Gamma_\epsilon \otimes id_A)^{\otimes N}$. Hence, we concentrate on the analysis of the estimation of $(\Gamma_\epsilon \otimes id_A)^{\otimes N}$. Figure 1 describes a typical example of the channel with $N = 2$. An entangled input state for the channel is generated from an initial factorized state by unitary operators U_1, U_2, and V. After the operation, each of the non-ancilla subsystems goes through a low-noise channel Γ_ϵ. The obtained output state is measured by a collective measurement M in order to estimate ϵ. The true low-noise parameter ϵ is generally small. Thus, only the analysis in the leading order of ϵ is important for applications. It has been found for the leading order of ϵ that the maximum value of the output Fisher information can be attained by a factorized input state for $(\Gamma_\epsilon \otimes id_A)^{\otimes N}$. This result is interesting because the number of characteristic parameters of the low-noise channels is very large.

2   Brief Review of Low-Noise Channels

In this section, we give a brief review of low-noise channels parametrized by the non-negative low-noise parameter ϵ [2]. The channels are defined in the Kraus representation as

$$\Gamma_\epsilon[\rho] = \sum_\alpha B_\alpha(\epsilon)\rho B_\alpha^\dagger(\epsilon) + \epsilon \sum_\beta C_\beta(\epsilon)\rho C_\beta^\dagger(\epsilon). \tag{4}$$

The Kraus operators must satisfy the following conditions.

(i) The channel is a trace-preserving completely positive (TPCP) map:

$$\sum_\alpha B_\alpha^\dagger(\epsilon)B_\alpha(\epsilon) + \epsilon \sum_\beta C_\beta^\dagger(\epsilon)C_\beta(\epsilon) = \mathbf{1}_S. \tag{5}$$

(ii) $B_\alpha(\epsilon)$ is analytic at $\epsilon = 0$, giving the power series expansion

$$B_\alpha(\epsilon) = \kappa_\alpha \mathbf{1}_S - \sum_{n=1}^\infty N_\alpha^{(n)} \epsilon^n \tag{6}$$

in the neighborhood of $\epsilon = 0$, where κ_α and $N_\alpha^{(n)}$ are coefficients and operators, respectively, independent of ϵ.

(iii) κ_α satisfies

$$\sum_\alpha |\kappa_\alpha|^2 = 1. \tag{7}$$

(iv) $C_\beta(\epsilon)$ is analytic at $\epsilon = 0$, giving the power series expansion

$$C_\beta(\epsilon) = M_\beta + \sum_{n=1}^\infty M_\beta^{(n)} \epsilon^n \tag{8}$$

where M_β and $M_\beta^{(n)}$ are operators independent of ϵ.

From condition (iii) (Eq. (7)), the channel automatically reduces to the identical channel in the noise-vanishing limit:

$$\lim_{\epsilon \to +0} \Gamma_\epsilon = id_S. \tag{9}$$

All physical channels should be TPCP maps and any TPCP map has Kraus representations. Thus condition (i) naturally applies to low-noise channels. Conditions (ii) and (iv) simply imply that the channel shows nonsingular behavior near $\epsilon = 0$. Therefore, taking proper limits of weak coupling with environment, physical processes induced by the environment can be always described by the low-noise channels.

For general low-noise channels, we can calculate perturbatively the output Fisher information corresponding to an input state $|\phi\rangle\langle\phi|$ [2]. The output Fisher information is evaluated in the leading order of ϵ as

$$J = \frac{1}{\epsilon} \sum_\beta \left[\langle\phi|M_\beta^\dagger M_\beta|\phi\rangle - |\langle\phi|M_\beta|\phi\rangle|^2 \right] + O(\epsilon^0), \tag{10}$$

where M_β is the lowest-order operator of $C_\beta(\epsilon)$ defined by Eq. (8). Note that this formula can be applied to any low-noise channel. Hence, in the next section, we may apply this formula to calculate the output Fisher information of an N-body extended low-noise channel.

We consider an ancillary system A and a composite system $S + A$. The low-noise channels are trivially extended as $\Gamma_\epsilon \otimes id_A$. For an input state $|\Psi\rangle\langle\Psi|$ of $\Gamma_\epsilon \otimes id_A$, the output Fisher information J_{S+A} is evaluated in the leading order as

$$J_{S+A} = \frac{1}{\epsilon} \sum_\beta \left[\mathrm{Tr}[\tilde{\rho} M_\beta^\dagger M_\beta] - |\mathrm{Tr}\,[\tilde{\rho} M_\beta]|^2 \right] + O(\epsilon^0), \tag{11}$$

where $\tilde{\rho}$ is a reduced state of S given by

$$\tilde{\rho} = \mathrm{Tr}_A[|\Psi\rangle\langle\Psi|]. \tag{12}$$

It should be stressed that $\tilde{\rho}$ can describe any possible state of the original system S.

3 $(\Gamma_\epsilon \otimes id_A)^{\otimes N}$ Channel Estimation

In general, a quantum channel Γ possesses a Kraus representation

$$\Gamma[\rho] = \sum_\alpha A_\alpha \rho A_\alpha^\dagger. \tag{13}$$

Using the Kraus representation, $\Gamma^{\otimes N}$ can be written as

$$\Gamma^{\otimes N}[\rho^{(N)}] = \sum_{\alpha_1 \alpha_2 \cdots \alpha_N} A_{\alpha_1} \otimes A_{\alpha_2} \otimes \cdots A_{\alpha_N} \rho^{(N)} A_{\alpha_1}^\dagger \otimes A_{\alpha_2}^\dagger \otimes \cdots A_{\alpha_N}^\dagger. \tag{14}$$

Similarly the Kraus operators of $\Gamma \otimes id_A$ can be derived to be $A_\alpha \otimes \mathbf{1}_A$. Thus the N-body extension of the low-noise channel $(\Gamma_\epsilon \otimes id_A)^{\otimes N}$ can be written as

$$(\Gamma_\epsilon \otimes id_A)^{\otimes N}[\rho^{(N)}] = \sum_{\alpha'} B_{\alpha'}^{(N)}(\epsilon) \rho^{(N)} B_{\alpha'}^{(N)\dagger}(\epsilon) + \epsilon \sum_{i\beta'} C_{i,\beta'}^{(N)}(\epsilon) \rho^{(N)} C_{i,\beta'}^{(N)\dagger}(\epsilon), \tag{15}$$

where the Kraus operators are expressed by

$$B_{\alpha'}^{(N)}(\epsilon) := B_{\alpha_1 \cdots \alpha_N}^{(N)}(\epsilon) = \left(\prod_{i=1}^N \kappa_i \right) \mathbf{1}^{(N)} \otimes \mathbf{1}_A^{(N)} + O(\epsilon), \tag{16}$$

$$C_{1,\beta'}^{(N)}(\epsilon) := C_{1,\beta_1\beta_2\cdots\beta_N}^{(N)}(\epsilon) = (M_{\beta_1} \otimes \mathbf{1}_A) \otimes (\kappa_{\beta_2}\mathbf{1}_S \otimes \mathbf{1}_A) \otimes \cdots (\kappa_{\beta_N}\mathbf{1}_S \otimes \mathbf{1}_A) + O(\epsilon), \tag{17}$$

$$C_{2,\beta'}^{(N)}(\epsilon) := C_{2,\beta_1\beta_2\cdots\beta_N}^{(N)}(\epsilon) = (\kappa_{\beta_1}\mathbf{1}_S \otimes \mathbf{1}_A) \otimes (M_{\beta_2} \otimes \mathbf{1}_A) \otimes \cdots (\kappa_{\beta_N}\mathbf{1}_S \otimes \mathbf{1}_A) + O(\epsilon) \tag{18}$$

and so on.

From Eq. (10), the output Fisher information for the input pure state $|\Psi^{(N)}\rangle\langle\Psi^{(N)}|$ can be evaluated in the leading order of ϵ as

$$J^{(N)} = \frac{1}{\epsilon} \sum_{i=1}^N \sum_{\beta'} \left[\langle\Psi^{(N)}|C_{i,\beta'}^\dagger C_{i,\beta'}|\Psi^{(N)}\rangle - \left| \langle\Psi^{(N)}|C_{i,\beta'}|\Psi^{(N)}\rangle \right|^2 \right] + O(\epsilon^0). \tag{19}$$

Using $\sum_{\beta_i} |\kappa_{\beta_i}|^2 = 1$, the expression can be simplified to

$$J^{(N)} = \frac{1}{\epsilon} \sum_{i=1}^N \sum_{\beta} \left[\langle\Psi^{(N)}|M_{i,\beta}^\dagger M_{i,\beta}|\Psi^{(N)}\rangle - \left| \langle\Psi^{(N)}|M_{i,\beta}|\Psi^{(N)}\rangle \right|^2 \right] + O(\epsilon^0), \tag{20}$$

6

where the operators $M_{i,\beta}$ are given by

$$M_{1,\beta} = (M_\alpha \otimes \mathbf{1}_A) \otimes (\mathbf{1}_S \otimes \mathbf{1}_A) \otimes \cdots (\mathbf{1}_S \otimes \mathbf{1}_A), \qquad (21)$$

$$M_{2,\beta} = (\mathbf{1}_S \otimes \mathbf{1}_A) \otimes (M_\alpha \otimes \mathbf{1}_A) \otimes \cdots (\mathbf{1}_S \otimes \mathbf{1}_A), \qquad (22)$$

and so on. Now we define a reduced state ρ_i^{S+A} at the i-th site:

$$\rho_i^{S+A} = \mathrm{Tr}_{[i]}[|\Psi^{(N)}\rangle\langle\Psi^{(N)}|], \qquad (23)$$

where $\mathrm{Tr}_{[i]}$ is the trace operation in terms of the $N-1$ sites except the i-th site. The output Fisher information can then be described by the reduced states as follows:

$$J^{(N)} = \frac{1}{\epsilon} \sum_{i=1}^{N} \sum_{\beta} \left[\mathrm{Tr}[\rho_i^{S+A} \left(M_\beta^\dagger \otimes \mathbf{1}_A \right) (M_\beta \otimes \mathbf{1}_A)] - \left| \mathrm{Tr}[\rho_i^{S+A} (M_\beta \otimes \mathbf{1}_A)] \right|^2 \right] + O(\epsilon^0).$$
$$(24)$$

By performing a trace operation on the ancilla system at the i-th site, we define a reduced state of the i-th subsystem as

$$\tilde{\rho}_i = \mathrm{Tr}_A[\rho_i^{S+A}]. \qquad (25)$$

Finally we obtain an expression for the output Fisher information:

$$J^{(N)} = \frac{1}{\epsilon} \sum_{i=1}^{N} \sum_{\beta} \left[\mathrm{Tr}[\tilde{\rho}_i M_\beta^\dagger M_\beta] - |\mathrm{Tr}[\tilde{\rho}_i M_\beta]|^2 \right] + O(\epsilon^0). \qquad (26)$$

Note that Eq. (26) is expressed by the site-sum ($\sum_i$) of independent contributions of the output Fisher information in Eq. (11). Hence, by simultaneously maximizing the output Fisher information at each site, $J^{(N)}$ trivially becomes maximum. We use the optimal input state $|\Psi_{opt}\rangle\langle\Psi_{opt}|$ for the single-system channel $\Gamma_\epsilon \otimes id_A$. The state $|\Psi_{opt}\rangle\langle\Psi_{opt}|$ maximizes the output Fisher information in Eq. (11). A factorized input state given by

$$|\Psi_{opt}^{(N)}\rangle\langle\Psi_{opt}^{(N)}| = (|\Psi_{opt}\rangle\langle\Psi_{opt}|)^{\otimes N} \qquad (27)$$

clearly gives maximum $J^{(N)}$. The reduced states $\tilde{\rho}_{i,opt}$ are given by

$$\tilde{\rho}_{i,opt} = \mathrm{Tr}_A[|\Psi_{opt}\rangle\langle\Psi_{opt}|], \qquad (28)$$

7

and maximize each site output Fisher information defined by Eq. (11). The maximum value is given by

$$J^{(N)}[|\Psi_{opt}^{(N)}\rangle\langle\Psi_{opt}^{(N)}|] = N J_{S+A}[|\Psi_{opt}\rangle\langle\Psi_{opt}|], \qquad (29)$$

where $J_{S+A}[|\Psi_{opt}\rangle\langle\Psi_{opt}|]$ is the output Fisher information for $\Gamma_\epsilon \otimes id_A$ with input $|\Psi_{opt}\rangle\langle\Psi_{opt}|$. Therefore the prior entanglement of the N subsystems cannot increase the Fisher information to larger than $N J_{S+A}[|\Psi_{opt}\rangle\langle\Psi_{opt}|]$.

This result implies that in order to attain an optimal estimation, it is sufficient to carefully arrange the entanglement between S and A, if the state $|\Psi_{opt}\rangle$ is an entangled state of $S + A$. It is not necessary to arrange entanglement of the N subsystems or make collective measurements over the subsystems.

Acknowledgement

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